ACTA CHEMICA SCANDINAVICA 24 (1970) 1800–1808

On Some Problems in the Symbolic Addition Procedure

JENS NYBORG

Department of Inorganic Chemistry, Chalmers Institute of Technology and the University of Göteborg, Fack, S-402 20, Göteborg 5, Sweden

It is the purpose of direct methods to determine the set of phases which will give the most probable solution of the crystal structure under investigation. A new program system which carefully follows every step of the symbolic addition procedure by probability calculations is described. A new way of solving relationships between symbols in non-centrosymmetric structures is introduced. This program system will under the limitations introduced in the applied probability formulas give the most probable solution of a crystal structure.

The symbolic addition procedure is described by Karle and Karle,[1,2] and is based on the works done by Sayre,[3] Cochran,[4] Zachariasen,[5] and Hauptman and Karle.[6] According to the theory of direct methods it is given that

$$\phi(\mathbf{h}) - (\phi(\mathbf{k}) + \phi(\mathbf{h}-\mathbf{k})) \approx 0 \text{ (modulo } 2\pi) \quad (1)$$

where $\phi(\mathbf{h})$ is the phase of the structure factor $F_{\mathbf{h}}$. In centrosymmetric space groups (1) can be formulated

$$s(\mathbf{h})s(\mathbf{k})s(\mathbf{h}-\mathbf{k}) \approx +1 \quad (2)$$

where $s(\mathbf{h})$ is the sign of the structure factor $F_{\mathbf{h}}$. The equality sign $\approx$ indicates that the equations are probably correct. The approximate probabilities that the equations are correct can be evaluated.

Eqns. (1) and (2) can be used for phase or sign determination. If two of the phases or signs are known the third one can be determined. Three (or less) phases or signs define the origin and can be chosen freely subject to space group limitations.[6] For a more effective use of (1) and (2) symbols for phases or signs can be assigned to a number of reflections.[1] New phases or signs can be determined with the help of these symbols.

In the following "phase" will be used for "phase or sign" and "symbol" for "symbol or combination of symbols".

The symbolic addition procedure is normally performed in cycles. In each cycle known phases or symbols are combined according to (1) or (2). When a phase is given as a combination of phases or symbols the phase is said to be indicated by this combination. Each cycle ends up with a set of indications of

some new phases. These indications can give a new set of known phases or symbols. The new set of phases or symbols are now used in a new cycle of the symbolic addition procedure. After several cycles many phases can be expressed by symbols and in most cases relationships between symbols will appear.[2]

In the last few years an increasing number of structures have been solved using the symbolic addition procedure and some attempts to automatize the procedure have been made. Even fairly complex structures have been solved in this way. Two main problems in using this systematic procedure have, however, been that in the first few cycles single indications of phases from (1) or (2) must be trusted and that relationships between the symbols must be accepted.[7]

One of the first steps in the symbolic addition procedure is the formation of the $\sum_2$-listing, which consists of all the combinations of $\mathbf{k}$ and $\mathbf{h}-\mathbf{k}$ for a given $\mathbf{h}$. The phase $\phi(\mathbf{h})$ is then determined by all phase combinations $\phi(\mathbf{k})$ and $\phi(\mathbf{h}-\mathbf{k})$ for $\mathbf{k}$'s occurring in this listing. If all the pairs of phases were known the phase $\phi(\mathbf{h})$ would normally be determined with a high probability. If on the other hand only one of the pair of phases is known the probability for the determination of the phase $\phi(\mathbf{h})$ is often relatively low and there is a great deal of risk in accepting the determination. As this is exactly the case in the first cycles of the symbolic addition procedure and as all the rest of the determinations are very much dependent on the correctness of these first cycles the whole phase determination can be wrong and the corresponding Fourier-maps cannot be interpreted in a reasonable manner. How wrong the determinations can go for complex structures with 200 or 400 atoms in the unit cell is very clearly pointed out by Germain and Woolfson.[7] They show that the maximal probability of a single indication of $s(\mathbf{h})$ in a centrosymmetric structure with 400 atoms in the unit cell is 0.881.

A new system of computer programs for solving crystal structures by the symbolic addition procedure has been written.[8] The indications of phases and the appearance of equations between symbols are followed closely by probability calculations in order to decide when indications of phases can be trusted and when relationships between symbols can be accepted. The selection of initial symbols, the probability calculations, and the solutions of relationships between symbols in this program system are described below.

SELECTION OF INITIAL SYMBOLS

Although the normal procedure for the selection of the initial symbols has been the following: Selection of three (or less) origin-defining phases and selection of a small number of symbols, the origin-defining phases are in this program system selected at the end of the symbolic addition.

The interrelationships between the phases are given by the $\sum_2$-listing and the assignment of symbols for phases is just a simple and useful way to obtain information from this listing. If symbols were assigned for all phases in the listing one would get another listing, a listing of symbols, which in symbolic form would be identical to the $\sum_2$-listing. In this new listing the symbol for

the phase $\phi(\mathbf{h})$ would be indicated by many different combinations of symbols, *i.e.* symbols for phases $\phi(\mathbf{k})$ and $\phi(\mathbf{h}-\mathbf{k})$ for different $\mathbf{k}$'s. The probability of a single of these indications would normally be relatively low. The probability of the symbol for the phase $\phi(\mathbf{h})$ could, however, be relatively high if some or all of the different indications were correct. This can happen only if there exist relationships between the different indications, *i.e.* relationships between the symbols are indicated.

The probability of a single indication of a phase is known and the probability of relationships between different indications can be calculated. A listing of indicated relationships between symbols can be formed. From indications for different phases different sets of indicated relationships will appear. In later cycles of the procedure a relationship will normally be indicated from several of these sets. The probability of this relationship will eventually be so high that it can be solved for one of the initial symbols. In this way the phases in the end of the procedure will be determined with the highest possible probability.

It is, however, impractical to assign symbols to all the phases. The new indicated phases will be very complicated combinations of symbols and the number of relationships between symbols will be enormous. The assignment is therefore given to a subset of the greatest structure factors. This subset should then be chosen great enough to give a reasonable amount of relationships with reasonable high probabilities.

Up to this point there has been no difference in the problems for centrosymmetric and non-centrosymmetric structures. In order to follow the procedure step for step it is, however, necessary to consider how to calculate the probabilities of the relationships and how to solve the relationships between symbols. As this is completely different for centrosymmetric and non-centrosymmetric structures these two cases are dealt with separately.

PROBABILITY CALCULATIONS

a) *Centrosymmetric structures.* In centrosymmetric space groups it is given that

$$sb(\mathbf{h}) \approx sb(\mathbf{k})sb(\mathbf{h}-\mathbf{k}) \tag{3}$$

with the approximate probability [9]

$$P_+(\mathbf{h}) = \tfrac{1}{2} + \tfrac{1}{2}\tanh(\sigma_2^{-3/2}\sigma_3|E_{\mathbf{h}}E_{\mathbf{k}}E_{\mathbf{h}-\mathbf{k}}|) \tag{3a}$$

where $sb(\mathbf{h})$ is the symbol for the sign of the normalized structure factor $E_{\mathbf{h}}$ and $\sigma_n = \sum_{j=1}^{N} Z_j^n$. If $\mathbf{h} = 2\mathbf{k}$ the approximate probability will be [9]

$$P_+(2\mathbf{k}) = \tfrac{1}{2} + \tfrac{1}{2}\tanh(\tfrac{1}{2}\sigma_2^{-3/2}\sigma_3|E_{2\mathbf{k}}|(E_{\mathbf{k}}^2 - 1))$$

$P_+(\mathbf{h})$ is a conditional probability, *i.e.* the probability of $sb(\mathbf{h})$ given by (3) when $sb(\mathbf{k})$ and $sb(\mathbf{h}-\mathbf{k})$ are known. In every cycle of the symbolic addition procedure new symbols are indicated from (3) and the known set of symbols. $P_+(\mathbf{h})$ can be evaluated for every new indication of a symbol. Nevertheless in

most circumstances the symbols $sb(\mathbf{k})$ and $sb(\mathbf{h}-\mathbf{k})$ are indicated from previous cycles with probabilities less than 1. Therefore the probability that the symbol $sb(\mathbf{h})$ is indicated from (3) is less than $P_+(\mathbf{h})$. Let the probability of $sb(\mathbf{k})$ be $p_\mathbf{k}$ and the probability of $sb(\mathbf{h}-\mathbf{k})$ be $p_{\mathbf{h}-\mathbf{k}}$. Then the probability of the product $sb(\mathbf{k})sb(\mathbf{h}-\mathbf{k})$ can be written:

$$P = p_\mathbf{k} p_{\mathbf{h}-\mathbf{k}} + (1-p_\mathbf{k})(1-p_{\mathbf{h}-\mathbf{k}})$$

Now the probability that $sb(\mathbf{h})$ is indicated by $sb(\mathbf{k})sb(\mathbf{h}-\mathbf{k})$ can be expressed by

$$p = P_+(\mathbf{h})P + (1-P_+(\mathbf{h}))(1-P)$$

p is then the total probability of a single indication of a symbol by the symbolic addition procedure.

In some cases there will be two or more independent indications of the symbol which are identical. The total probability, p_s, will then normally be greater than any one of the independent probabilities $p_1, p_2, p_3, \ldots$

$$p_s = \frac{p_1 p_2 p_3 \ldots}{p_1 p_2 p_3 \ldots\ldots + (1-p_1)(1-p_2)(1-p_3)\ldots\ldots}$$

Only indications with sufficiently high values of p_s are accepted.

The symbol $sb(\mathbf{h})$ can also be indicated by two or more different symbols each with probabilities $p_s^1, p_s^2, p_s^3, \ldots\ldots$ Relationships between symbols different from (3) are thus indicated. The probability of the relationship

$$sb(\mathbf{k}_1)sb(\mathbf{h}-\mathbf{k}_1) \approx sb(\mathbf{k}_2)sb(\mathbf{h}-\mathbf{k}_2) \qquad (4)$$

is, *e.g.*,

$$p_{\mathrm{L}'} = p_s^1 p_s^2 + (1-p_s^1)(1-p_s^2)$$

There may again exist independent indications of this relationship giving the higher probability

$$p_{\mathrm{tot}} = p_{\mathrm{L}'}{}^a p_{\mathrm{L}'}{}^b \ldots\ldots / (p_{\mathrm{L}'}{}^a p_{\mathrm{L}'}{}^b \ldots\ldots + (1-p_{\mathrm{L}'}{}^a)(1-p_{\mathrm{L}'}{}^b)\ldots)$$

where $p_{\mathrm{L}'}{}^a, p_{\mathrm{L}'}{}^b, \ldots\ldots$ are the probabilities of the independent indications.

Only relationships with high values of p_{tot} are solved. Hereby one of the initial symbols can be expressed as a product of other initial symbols with the probability p_{tot}. It can now be eliminated from both the set of phase indications and the set of relationship indications and the probabilities are modified as

$$p_\mathrm{n} = p_{\mathrm{tot}} p_\mathrm{o} + (1-p_{\mathrm{tot}})(1-p_\mathrm{o})$$

where p_o is the probability of an indication of a phase or a relationship before the elimintaion and p_n is the probability after.

b) *Non-centrosymmetric structures.* In non-centrosymmetric structures the useful relationship is

$$sb(\mathbf{h}) \approx sb(\mathbf{k}) + sb(\mathbf{h}-\mathbf{k}) + n_{\mathbf{h},\mathbf{k}} \cdot 2\pi \qquad (5)$$

where $n_{\mathbf{h},\mathbf{k}}$ is an integer. The relationship is given with the approximate probability distribution for fixed phases $\phi(\mathbf{k})$ and $\phi(\mathbf{h}-\mathbf{k})$ [10,2]

$$P(\phi(\mathbf{h})) = (2\pi I_0(\varkappa))^{-1} \exp(\varkappa\cos(\phi(\mathbf{h}) - \phi(\mathbf{k}) - \phi(\mathbf{h}-\mathbf{k}))) \quad (5a)$$

where $\varkappa = 2\sigma_2^{-3/2}\sigma_3|E_\mathbf{h}E_\mathbf{k}E_{\mathbf{h}-\mathbf{k}}|$ and I_0 is the hyperbolic Bessel function of order 0. As none of the phases are known it is more convenient to evaluate the variance of the phase $\phi(\mathbf{h})$ (Ref. 2).

$$V(\mathbf{h}) = \pi^2/3 + 4(I_0(\varkappa))^{-1} \sum_{j=0}^{\infty} (-1)^j I_j(\varkappa)/j^2 \quad (5b)$$

I_j is the hyperbolic Bessel function of order j.

New symbols can be indicated from (5) and the known set of symbols and for every such indication $V(\mathbf{h})$ can be evaluated. This is, however, the variance for fixed values of the phases $\phi(\mathbf{k})$ and $\phi(\mathbf{h}-\mathbf{k})$. As these are indicated by $sb(\mathbf{k})$ and $sb(\mathbf{h}-\mathbf{k})$ which have variances normally greater than 0 the single indication (5) has a variance greater than $V(\mathbf{h})$. Let the variance of $sb(\mathbf{k})$ be $var_\mathbf{k}$ and the variance of $sb(\mathbf{h}-\mathbf{k})$ be $var_{\mathbf{h}-\mathbf{k}}$. Then the variance of the sum $sb(\mathbf{k}) + sb(\mathbf{h}-\mathbf{k})$ is

$$V = var_\mathbf{k} + var_{\mathbf{h}-\mathbf{k}}$$

and the variance of a single indication of $sb(\mathbf{h})$ by $sb(\mathbf{k}) + sb(\mathbf{h}-\mathbf{k})$

$$var = V(\mathbf{h}) + V.$$

As for the centrosymmetric structures there will in some cases exist two or more independent indications of the same symbol. The variance of the indication will then be less than *var:*

$$1/var_s = 1/var_1 + 1/var_2 + 1/var_3 + \ldots$$

where var_1, var_2, var_3 ... are variances of the independent indications. Only symbols with sufficiently low value of *var* are accepted. The symbol $sb(\mathbf{h})$ can be indicated by two or more different symbols each with variances var_s^1, var_s^2, var_s^3, ... Relationships of the form

$$sb(\mathbf{k}_1) + sb(\mathbf{h}-\mathbf{k}_1) \simeq sb(\mathbf{k}_2) + sb(\mathbf{h}-\mathbf{k}_2) + (n_{\mathbf{h},\mathbf{k}_1} - n_{\mathbf{h},\mathbf{k}_2})2\pi \quad (6)$$

can be formed. The variance of the indication of such a relationship can be evaluated, *e.g.*

$$var_L = var_s^1 + var_s^2$$

Relationships which are identical can be indicated from different phase indications. This will give an indication of that relationship with a lower variance:

$$1/var_{tot} = 1/var_L^a + 1/var_L^b + \ldots$$

where var_L^a, var_L^b, ... are variances of the independent indications.

Relationships with low values of var_{tot} are solved for one of the initial symbols. This symbol is eliminated from the set of phase indications and the set of relationship indications and the variances are modified

$$var_n = var_{tot} + var_o$$

where var_o is the variance before, var_n the variance after the elimination.

SOLUTIONS OF RELATIONSHIPS BETWEEN SYMBOLS

a) *Centrosymmetric structures.* The relationships between symbols are all of the form

$$s_1 s_2 s_3 \ldots = +1$$

where s_1, s_2, s_3, ... are initial symbols. As $s_i^2 = +1$ no initial symbol occurs more than one time in the same relationship. A relationship can be solved for any one of the symbols s_1, s_2, s_3, ...

b) *Non-centrosymmetric structures.* The relationships between symbols are here of the form

$$n_1 s_1 + n_2 s_2 + n_3 s_3 + \ldots = n_0 2\pi$$

where n_0, n_1, n_2, n_3, ... are integers. The initial symbols can occur several times in the same relationship because in this case there is no relation which limits the n's. Now the solution of such a relationship must give a result with a period of 2π. Therefore the relationship can be solved for an initial symbol if this symbol occurs only one time in this relationship, *i.e.* the corresponding $n = +1$ or -1.

DETERMINATION OF PHASES

In the last cycle of the symbolic addition procedure the origin-defining phases are selected. Usually not all the initial symbols are solved in terms of other symbols and this will give the possibility of several different solutions. The different sets of possible determinations of phases are used in subsequent refinements of the phases.

a) *Centrosymmetric structures.* If there are n_u undetermined initial symbols there will be 2^{n_u} different possible solutions. Each of the solutions is refined using the $\sum_2$-formula [6]

$$s(\mathbf{h}) \approx s \sum_{\mathbf{k}} E_{\mathbf{k}} E_{\mathbf{h}-\mathbf{k}} \tag{7}$$

where s means the sign of the sum. The approximate probability of this sign determination [9]

$$P_+(\mathbf{h}) = \tfrac{1}{2} + \tfrac{1}{2}\tanh(\sigma_2^{-3/2}\sigma_3 |E_{\mathbf{h}}| \sum E_{\mathbf{h}} E_{\mathbf{h}-\mathbf{k}})$$

is calculated and the number of positive, negative and undetermined signs (signs for which $P_+(\mathbf{h})$ is neither big nor small) are counted. Solutions with the smallest number of undetermined signs are believed to be the most correct solutions and are used for Fourier summations. Another criterion is that the number of positive and negative signs must be almost equal if no atom is supposed to be placed at the origin.

b) *Non-centrosymmetric structures.* The relationships which could not be solved can now be used to determine a number of possible solutions for the n_u undetermined initial symbols. If it is possible n_u linearly independent relationships are solved for the undetermined initial symbols in terms of the integers n_0. This will give a finite number of possible solutions. To illustrate this an example is given.[11] The two symbols d and k are the only undetermined symbols.

The following two relationships exist between the symbols:

$$4k = p \cdot 2\pi$$
$$2d + 3k = t \cdot 2\pi$$

where p and t are integers. The solution of these two relationships will give

$$k = \frac{p}{2} \cdot \pi$$

$$d = \left(t - \frac{3p}{4}\right) \cdot \pi$$

One of the phases can be used to fix the enanthiomorph. The possible solutions to these two equations are given in Table 1. Each of the possible solutions are refined using the tangent formula:[2]

$$\tan\phi(\mathbf{h}) = \frac{\sum_{\mathbf{k}} E_{\mathbf{k}} E_{\mathbf{h}-\mathbf{k}} \sin(\phi(\mathbf{k}) + \phi(\mathbf{h}-\mathbf{k}))}{\sum_{\mathbf{k}} E_{\mathbf{k}} E_{\mathbf{h}-\mathbf{k}} \cos(\phi(\mathbf{k}) + \phi(\mathbf{h}-\mathbf{k}))} \tag{8}$$

and the corresponding R_E-value [2] is calculated.

$$R_E = \frac{\sum |E_{obs} - E_{calc}|}{\sum |E_{obs}|}$$

where E_{calc} is a calculated and E_{obs} an observed normalized structure factor. Solutions with the lowest R_E-values are believed to be most correct and are used for Fourier-calculations. As an example see Table 1 where set No. 2 gave the correct structure.

Table 1. The four possible sets of phases.

Set	k	d	R_E in %
1	0	0	22.85
2	$\pi/2$	$-3\pi/4$	20.34
3	$\pi/2$	$\pi/4$	22.14
4	π	$-\pi/2$	21.39

CONCLUSIONS

There are yet many unsolved problems in the symbolic addition procedure. There are problems as well in the theory of direct methods as in the practical application of symbolic addition especially in the automatization of the procedure.

The probability calculations have so far been based on the assumption of an even distribution of the atoms in the unit cell. Only a few scientists have worked with uneven distributions, *i.e.* insertion of known positions of some atoms, insertion of known interatomic distances *etc.* (*e.g.* Bertaut [12]). In this paper the probability formulas (3a) and (5a) are approximate and more accurate formulas could have been used (Naya, Nitta and Oda;[13] Danielsen [14]).

Further probability calculations in this paper are given even on the assumption that all single indications of phases are statistically independent. This is not necessarily true. As the probability calculations are essential in every application of symbolic addition there is still a lot of work to be done in the evaluation of better probability distributions.

One of the drawbacks of the procedure described in this paper is that the probability calculations also are based upon space groups $P1$ and $P\bar{1}$. All space groups of higher symmetry are therefore to be considered as space groups $P1$ or $P\bar{1}$ with some relations between the phases or signs arising from the symmetry. Therefore it is especially desirable not to trust upon the automatically determined phases of special reflections such as plane and axial reflections. An example is plane reflections in space group $P2_12_12_1$ where the application of formula (1) to such reflections can give wrong answers.

There are also problems in the automatization of the symbolic addition in the space group $P\bar{1}$ itself. The relation (2) will for the greatest structure factors give all signs equal to $+1$. If the space group was considered to be $P1$ all the phases would be zero. But if now the origin is shifted from the inversion center the phases will not all be equal; *i.e.* the phases in the new system are related to the old ones not only by the shift in origin but also by the length of the scattering vector. The solution of relationships between phases and subsequent refinement by the tangent formula (8) will give phases which are more dependent of the magnitudes of the structure factors than are the signs determined in space group $P\bar{1}$ and refined by the $\sum_2$-formula (7). One crystal structure in space group $P\bar{1}$ has been solved in this way.[11]

Another serious problem in the application of symbolic addition is that the probability values are very much dependent of the correctness of the observed structure factors. It is known that in every structure determination some of the observed structure factors have magnitudes which can be far from the right ones. This is certainly the case for reflections near the rotation axis where *Lp*-factors are very great. Nowadays it is possible to make very accurate measurements on crystal diffraction. It would therefore be desirable to have a method by which one could check the correctness of the observed structure factors.

The ambiguity problem described by Karle and Karle [2] (p. 858) is solved in this paper. It arises from the fact that the phase determining relation

$$\phi(\mathbf{h}) = \langle \phi(\mathbf{k}) + \phi(\mathbf{h}-\mathbf{k}) \rangle_{\mathbf{k}}$$

is correct only under the assumption that all $n_{\mathbf{h},\mathbf{k}}$'s are equal for all pairs of phases in the averaging. This is not necessarily the case and can give rise to some confusion about the correct determination of the phases. This formula should therefore not be used in structure determinations.

There is no doubt that the symbolic addition procedure is very useful in determining crystal structures. But so far most structures have been solved using either full or at least some hand-application of the procedure. Very little is up till now known about a fully automatized procedure. Recently two new program systems have been described by Germain and Woolfson [7] and Danielsen.[14] They are both based on multiple solution principle and seem to be rather successful. The reason why they use this principle is, however, that they

restrict themselves to indications of the type (1) or (2). As mentioned before this can give rather low probabilities of correct determination of phases especially for large structures. Introduction of additional indications of the type (4) and the probability calculations in this paper can, however, raise this probability and the solution of large crystal structures should therefore not be so difficult as described by Germain and Woolfson.[7]

Acknowledgements. I wish to thank Professor G. Lundgren for inviting me to the University of Göteborg, fil. mag. O. Lindgren and fil. lic. O. Lindqvist for introducing me to the IBM-360/50 system and for many helpful discussions, and dr. phil. J. Danielsen, University of Aarhus, for teaching me direct methods and for many valuable and interesting discussions.

This work has been supported by the *Swedish Natural Science Research Council* (contract No. 2318) and by *Chalmers Institute of Technology*, which has supplied a grant to cover the costs of the computer work at *Göteborg University's Computing Centre.*

REFERENCES

1. Karle, I. L. and Karle, J. *Acta Cryst.* **16** (1963) 969.
2. Karle, J. and Karle, I. L. *Acta Cryst.* **21**(1966) 849.
3. Sayre, D. *Acta Cryst.* **5** (1952) 60.
4. Cochran, W. *Acta Cryst.* **5** (1952) 65.
5. Zachariasen, W. H. *Acta Cryst.* **5** (1952) 68.
6. Hauptman, H. and Karle, J. *Solution of the Phase Problem. I. The Centrosymmetric Crystal,* A.C.A. Monograph No. 3, Pittsburgh: Polycrystal Book Service.
7. Germain, G. and Woolfson, M. M. *Acta Cryst.* **B 24** (1968) 91.
8. Lindgren, O., Lindqvist, O. and Nyborg, J. *Acta Chem. Scand.* **24** (1970) 732.
9. Cochran, W. and Woolfson, M. M. *Acta Cryst.* **8** (1955) 1.
10. Cochran, W. *Acta Cryst.* **8** (1955) 473.
11. Nyborg, J. and Danielsen, J. *Acta Chem. Scand.* **24** (1970) 59.
12. Bertaut, E. F. *Acta Cryst.* **11** (1958) 405.
13. Naya, S., Nitta, I. and Oda, T. *Acta Cryst.* **19** (1965) 734.
14. Danielsen, J. *Thesis,* University of Aarhus 1969.

Received November 29, 1969.